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Hamiltonian jump graphs

Baoyindureng Wu[a, b], Jixiang Meng[a]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, P.R.China
[b] Institute of Systems Science, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, P.R. China

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Abstract

Let G be a nonempty graph. The jump graph $J(G)$ of G is the graph whose vertices are edges of G, and where two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G. Equivalently, the jump graph $J(G)$ of G is the complement of line graph $L(G)$ of G. In this paper, we characterize hamiltonian jump graphs and settle two conjectures posed by Chartrand et al. on jump graphs.

Keywords: Jump graph; Line graph; Hamilton cycle

1. Introduction

All graphs considered here are finite, undirected and simple. We refer to [2] for unexplained terminology and notation. Let $G=(V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the order and the size of G, respectively. For a vertex v of G, if there is no confusion, the degree $d_G(v)$ and the neighborhood $N_G(v)$ of v are simply denoted by $d(v)$ and $N(v)$, respectively. Also, $I_G(v)$ denotes the set of all edges incident with v, then $|I_G(v)|=d_G(v)$. The symbols $\Delta(G)$, $\delta(G)$, $\omega(G)$, $\kappa(G)$ and $\alpha(G)$ denote the maximum degree, the minimum degree, the number of components, the connectivity, and the independence number of G, respectively. The closure of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains (where p is the order of G). We denote the closure of G by $cl(G)$. It is known from [1] that $cl(G)$ is

E-mail addresses: baoyin@amss.ac.cn (B. Wu), mjx@xju.edu.cn (J. Meng).

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well-defined, and G is hamiltonian if and only if $cl(G)$ is hamiltonian. Hence, if $cl(G)$ is complete, then G is hamiltonian.

As usual, P_n, C_n and K_n are, respectively, the path, cycle, and complete graph of order n. $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices. In particular, $K_{1,s}$ is called a star. K_n^- denotes the graph resulting from K_n by deleting an edge while K_n^+ the graph by adding a new vertex and joining it to exactly one vertex of K_n. The graph $K_3 \circ K_1$ is obtained from K_3 by adding for every vertex x of K_3 some new vertex y_x only adjacent to x. We say two graphs G and H are disjoint if they have no vertex in common, and denote their union by $G + H$; it is called the disjoint union of G and H. The disjoint union of k copies of G is written as kG.

The line graph $L(G)$ of G has the edges of G as its vertices and two vertices of $L(G)$ are adjacent if and only if they are adjacent in G. We call the complement of line graph $L(G)$ the jump graph $J(G)$ of G, followed by Chartrand et al. [3]. That is, the jump graph $J(G)$ is the graph defined on $E(G)$, and in which two vertices are adjacent if and only if they are not adjacent in G. Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G, and isolated vertices of G (if G has) play no role in line graph transformation and jump graph transformation, we assume that the graph G under consideration is nonempty and has no isolated vertices in what follows.

Harary and Nash-Williams [5] showed that the line graph $L(G)$ of a graph G is hamiltonian if and only if $G \cong K_{1,n}$, where $n \geqslant 3$, or G contains a dominating circuit. (A circuit C in G is dominating if every edge of G is incident with a vertex of C.) In this paper, we deal with a problem: when jump graphs are hamiltonian? In [3], Chartrand et al. presented some sufficient conditions for a jump graph to be hamiltonian. In addition, they posed the following two conjectures.

Conjecture A. Let G be a graph of order at least 7 and size $q \geqslant 5$. If $q \geqslant 2\Delta(G)$, then $J(G)$ is hamiltonian.

Conjecture B. Let G be a hamiltonian graph of order $p \geqslant 7$ and size q. If $q \geqslant 2p - 2$, then $J(G)$ is hamiltonian.

We shall give a nice characterization of hamiltonian jump graphs. As a corollary, Conjecture B is confirmed, and Conjecture A is disproved. Our main result is as follows.

Theorem 1.1. *Let G be a graph of size $q \geqslant 1$. Then $J(G)$ is not hamiltonian if and only if G satisfies one of the following conditions*:

(1) $q < 2\Delta(G)$.
(2) $q = 2\Delta(G)$ *and G has an edge uv such that* $d(u) = d(v) = \Delta(G)$.
(3) *G is isomorphic to* $K_3 + P_3$, $K_3 + 2K_2$ *or* $C_4 + K_2$.
(4) *G has a subgraph isomorphic to*

$$\begin{cases} K_3^+ & & & if\ q = 6,\\ K_4^- & or & K_3 \circ K_1 & if\ q = 7,\\ K_4 & & & if\ q = 8.\end{cases}$$

(5) *G is isomorphic to* K_5.

Since the size of any graph described in (3)–(5) of Theorem 1.1 is not greater than 10, we have

Theorem 1.2. *Let G be a graph of size $q \geqslant 11$. Then $J(G)$ is hamiltonian if and only if $q > 2\Delta(G)$ or $q = 2\Delta(G)$ and G has no edge uv such that $d(u) = d(v) = \Delta(G)$.*

Applying Theorem 1.2, we show that Conjecture B is true.

Corollary 1.3. *Let G be a hamiltonian graph of order $p \geqslant 7$ and size q. If $q \geqslant 2p - 2$, then $J(G)$ is hamiltonian.*

Proof. Since $q \geqslant 2p-2$ and $p-1 \geqslant \Delta(G)$, we have $q \geqslant 2\Delta(G)$. Moreover, by $p \geqslant 7$, $q \geqslant 12$. Suppose, on the contrary, that the result is false, and G is a counterexample. Then it follows from Theorem 1.2 that $q = 2\Delta(G)$, and G has an edge uv such that $d(u) = d(v) = \Delta(G)$. On the other hand, since G has a hamiltonian cycle, there exist at least two edges of G that are not adjacent to uv in G. Thus,

$$q \geqslant 1 + (d(u) - 1) + (d(v) - 1) + 2 = 2\Delta(G) + 1 = q + 1,$$

a contradiction. $\square$

To disprove Conjecture A, let G be a graph satisfying (2) in Theorem 1.1, with additional property that $\Delta(G) \geqslant 6$. Then G satisfies the assumption of Conjecture A, but $J(G)$ is not hamiltonian.

2. Preliminaries

We recall two classical results on hamiltonian graph theory. The first one is due to Ore [6], and the second to Chvátal and Erdös [4].

Theorem 2.1. *Let G be a graph of order $p \geqslant 3$. If $d_G(u) + d_G(v) \geqslant p$ for each pair of nonadjacent vertices u and v in G, then G is hamiltonian.*

Theorem 2.2. *If $\alpha(G) \leqslant \kappa(G)$ for a graph G, then G is hamiltonian.*

Let G be a graph of size q and $H = J(G)$. Then the order of H is q and $d_H(e) = q - d(u) - d(v) + 1$ for a vertex $e = uv \in V(H)$. For an edge e of G, we denote $d_{J(G)}(e)$ simply by $d_J(e)$. The following is similar to Theorem 9 in [3].

Corollary 2.3. *Let G be a graph of size $q \geqslant 3$. If $\Delta(G) \leqslant \frac{q+2}{4}$, then $J(G)$ is hamiltonian.*

Proof. Let e and e' be any two nonadjacent vertices of $J(G)$, which, in turn, are two adjacent edges of G. Let $e = uv$ and $e' = vw$. Then

$$\begin{aligned} d_J(e) + d_J(e') &= (q - d(u) - d(v) + 1) + (q - d(v) - d(w) + 1) \\ &\geqslant 2q - 4\Delta(G) + 2 \\ &\geqslant q. \end{aligned}$$

Since $J(G)$ has the order of q, and e, e' are chosen arbitrarily, Theorem 2.1 implies that $J(G)$ is hamiltonian. □

Next, we will study the independence number and connectivity of jump graphs.

Theorem 2.4. *Let G be a graph, then*

$$\alpha(J(G)) = \begin{cases} 3 & if\ \Delta(G) = 2 \quad and\ G\ has\ a\ triangle\ K_3, \\ \Delta(G) & otherwise. \end{cases}$$

Proof. First note that for a vertex $v \in V(G)$, $I_G(v)$ is an independent set of $J(G)$, and since $|I_G(v)| = d(v)$, we have $\alpha(J(G)) \geqslant \Delta(G)$. Secondly, it is easy to see that if $S \subseteq E(G)$ is an independent set of $J(G)$, then $G[S]$ is isomorphic to $K_{1,3}$ or K_3 if $|S| = 3$, otherwise, is isomorphic to $K_{1,r}$, where $r = |S|$. Thus the result is immediate. □

For a graph G, we denote $\max\{d(u) + d(v) \mid uv \in E(G)\}$ simply by $\xi(G)$. It is obvious that for any graph G of size q, we have $q \geqslant \xi(G) - 1$, and the equality holds if and only if G contains an edge e that is adjacent to every other edge of G. In [3], Chartrand et al. proved that for a graph of order at least 5, $J(G)$ is connected if and only if G contains no edge that is adjacent to every other edge of G. So, it can equivalently formulated as follows: for a graph of order at least 5 and size q, $J(G)$ is connected if and only if $q \geqslant \xi(G)$. On the other hand, one can easily check that in all graphs of order no more than 4, C_4 and K_4 are the only two graphs G such that $|E(G)| \geqslant \xi(G)$ and $J(G)$ are not connected. Note that $|E(C_4)| = 4 = \xi(C_4)$, $|E(K_4)| = 6 = \xi(K_4)$,$J(C_4) \cong 2K_2$ and $J(K_4) \cong 3K_2$. Hence, we have the following lemma.

Lemma 2.5. *For a graph G of size $q \geqslant 2$, $J(G)$ is connected if and only if $G \notin \{C_4, K_4\}$ and $q \geqslant \xi(G)$.*

In the proof of the following theorem, we shall use two basic facts: for any $G' \subseteq G$, we have $\xi(G') \leqslant \xi(G)$, $J(G') \subseteq J(G)$, and $J(G) - E' = J(G - E')$ for any $E' \subseteq E(G)$.

Theorem 2.6. *If $J(G)$ is connected for a graph G, then $\kappa(J(G)) \geqslant \delta(J(G)) - 1$ and the equality holds if and only if one of the following conditions is satisfied*:

(1) $\xi(G) = 4$ *and G contains* C_4.
(2) $\xi(G) = 6$ *and G contains* K_4.

Proof. Let E' be an arbitrary subset of $E(G)$ with $|E'| = \delta(J(G)) - 2$. To prove $\kappa(J(G)) \geqslant \delta(J(G)) - 1$, it suffices to show that $J(G) - E'$ is connected. Since $J(G) - E' = J(G - E')$, we consider $G-E'$. Since $\delta(J(G))=|E(G)|-\xi(G)+1$, we have $|E(G-E')|=\xi(G)+1\geqslant\xi(G-E')+1$. It implies that $G-E'\not\cong C_4$ or K_4 (even we neglect the possible isolated vertices of $G-E'$). Hence $J(G-E')$ is connected by Lemma 2.5.

Now we assume that $\kappa(J(G))=\delta(J(G))-1$. Let $E'\subseteq E(G)$ be a minimal vertex cut of $J(G)$. Then $|E'|=\delta(J(G))-1$, and by the similar argument as above, we have $|E(G-E')|=\xi(G)\geqslant\xi(G-E')$. Since $J(G-E')$ is not connected, by Lemma 2.5 we conclude that $|E(G-E')|=\xi(G-E')$, and further that $G-E'$ has only one nontrivial component that is isomorphic to C_4 or K_4. Hence, if $G-E'$ contains C_4, then G contains C_4 and $\xi(G)=\xi(G-E')=\xi(C_4)=4$; if $G-E'$ contains K_4, then G contains K_4 and $\xi(G)=\xi(G-E')=\xi(K_4)=6$. Conversely, suppose that G satisfies (1), and let H be a subgraph isomorphic to C_4. Since $\xi(G)=4$, H is a component of G. Let $E''=E(G)\backslash E(H)$, then $J(G)-E''=J(G-E'')=J(H)\cong J(C_4)$ is not connected and $|E''|=|E(G)|-4=|E(G)|-\xi(G)=\delta(J(G))-1$. Thus, $\kappa(J(G))=\delta(J(G))-1$. If G satisfies (2), then we similarly prove that $\kappa(J(G))=\delta(J(G))-1$. $\square$

Corollary 2.7. *Let G be a graph of size q and* $\Delta(G)\geqslant 4$. *If* $q\geqslant 3\Delta(G)-1$, *then* $J(G)$ *is hamiltonian.*

Proof. We have $\kappa(J(G))=\delta(J(G))$ or $\delta(J(G))-1$ from Theorem 2.6, and $\alpha(J(G))=\Delta(G)$ from Theorem 2.4, respectively. Note that $\delta(J(G))=q-\xi(G)+1$, $\xi(G)\leqslant 2\Delta(G)$ and $q\geqslant 3\Delta(G)-1$. If $\kappa(J(G))=\delta(J(G))$, then

$$\kappa(J(G))=q-\xi(G)+1\geqslant 3\Delta(G)-1-2\Delta(G)+1=\Delta(G)=\alpha(J(G)).$$

Thus, $J(G)$ is hamiltonian by Theorem 2.2. If $\kappa(J(G))=\delta(J(G))-1$, then $\xi(G)\leqslant 6$ by Theorem 2.6. Hence, we have $\kappa(J(G))=q-\xi(G)\geqslant 3\Delta(G)-7>\Delta(G)$, since $\Delta(G)\geqslant 4$. Once again by Theorem 2.2, $J(G)$ is hamiltonian. $\square$

Lemma 2.8. *Let G be a graph of size q. Then* $J(G)$ *is not hamiltonian if* $q<2\Delta(G)$, *or* $q=2\Delta(G)$ *and G has an edge* $uv\in E(G)$ *such that* $d(u)=d(v)=\Delta(G)$.

Proof. First, assume that $q<2\Delta(G)$ and let $u\in V(G)$ with $d(u)=\Delta(G)$. Let $S=E(G)\backslash I_G(u)$, then we have $\omega(J(G)-S)=|I_G(u)|=\Delta(G)>|S|$. So, $J(G)$ is not hamiltonian. Now, assume that $q=2\Delta(G)$ and let $e=uv\in E(G)$ be an edge such that $d(u)=d(v)=\Delta(G)$. Then, $d_J(uv)=2\Delta(G)-\Delta(G)-\Delta(G)+1=1$, and thus $J(G)$ is not hamiltonian. $\square$

3. Jump graph of G with small $\Delta(G)$

Lemma 3.1. *Let G be a graph of size q and* $\Delta(G)=1$. *Then* $J(G)$ *is not hamiltonian if and only if* $q\leqslant 2$.

Proof. The result is trivial, since $\Delta(G)=1$ implies that $J(G)\cong K_q$. $\square$

Before proceeding, we need a definition. Let G be a graph. We say that a graph F is obtained from G by splitting a vertex v of G if F is obtained from G by replacing a vertex v of G by two vertices v_1 and v_2 such that each neighbor of v in G is adjacent to exactly one of v_1 and v_2 in F. Clearly, $J(G)$ is a spanning subgraph of $J(F)$. Thus, if $J(G)$ is hamiltonian, so is $J(F)$.

Lemma 3.2. *Let G be a graph of size q and* $\Delta(G)=2$. *Then* $J(G)$ *is not hamiltonian if and only if one of the following holds*:

(1) $q<4$.
(2) $q=4$ *and G has an edge uv such that* $d(u)=d(v)=2$.
(3) *G is isomorphic to* K_3+P_3, K_3+2K_2 *or* C_4+K_2.

Proof. By Lemma 2.8, $J(G)$ is not hamiltonian if G satisfies (1) or (2). If G satisfies (3), it is easy to verify that $J(G)$ is not hamiltonian.

Now we assume that G does not satisfy any condition of the lemma. It suffices to show that $J(G)$ is hamiltonian. If $q\geqslant 6$, then by Corollary 2.3, $J(G)$ is hamiltonian. If $q=4$, then $G\cong 2P_3$ or P_3+2K_2. Clearly both $J(2P_3)\cong C_4$ and $J(P_3+2K_2)\cong K_4^-$ have a Hamilton cycle. If $q=5$ and G contains no C_3 or C_4, then $G\cong C_5$ or can be obtained from C_5 by splitting some vertices of C_5. Since $J(C_5)\cong C_5$, $J(G)$ is hamiltonian. □

Lemma 3.3. *Let G be a graph of size* $q\geqslant 6$ *and* $\Delta(G)=3$. *Then* $J(G)$ *is not hamiltonian if and only if* $q<9$ *and G contains a subgraph isomorphic to*

$$\begin{cases} K_3^+ & if\ \ q=6,\\ K_4^- \ \ or\ \ K_3\circ K_1 & if\ \ q=7,\\ K_4 & if\ \ q=8.\end{cases}$$

Proof. First suppose that $q=6$ and G contains a subgraph H isomorphic to K_3^+. Let $E'=E(G)\backslash E(H)$, then $|E'|=2$. Since $J(G)-E'=J(G-E')=J(H)$, and $J(K_3^+)\cong 2K_1+K_2$, we have $\omega(J(G)-E')=3>2=|E'|$, and thus $J(G)$ is not hamiltonian. Note that $J(K_4^-)\cong K_1+2K_2$ and $J(K_4)\cong 3K_2$. Therefore, we can similarly prove that $J(G)$ is not hamiltonian if $q=7$ and G has a subgraph $H\cong K_4^-$, or $q=8$ and G has a subgraph $H\cong K_4$. Now let us consider the case that $q=7$ and G contains a subgraph H isomorphic to $K_3\circ K_1$. Since the size of $K_3\circ K_1$ is 6, G is isomorphic to $(K_3\circ K_1)+K_2$ or can be (vertex-)split into $(K_3\circ K_1)+K_2$. It is easy to see that $J((K_3\circ K_1)+K_2)$ is not hamiltonian, and so $J(G)$ is not hamiltonian.

Now we assume that G does not satisfy any condition of the lemma. We shall show that $J(G)$ is hamiltonian. If $q\geqslant 9$, or $q=8$ and G contains no subgraph isomorphic to K_4, then by Theorem 2.6, $\kappa(J(G))\geqslant 3$, and by Theorem 2.4, $\alpha(J(G))=3$. Hence, by Theorem 2.2, $J(G)$ is hamiltonian. Now suppose that $q=6$. Let u be a vertex with $d(u)=3$, and $I_G(u)=\{e_1,e_2,e_3\}$. We label the remaining three edges of G as e_1', e_2', e_3', respectively. Since G has no subgraph isomorphic to K_3^+, we can further assume that e_i and e_j' are not adjacent in G when $i\neq j$. Then by our labelling, $e_ie_j'\in E(J(G))$ if $i\neq j$, so $J(G)$ has a Hamilton cycle: $e_1e_2'e_3e_1'e_2e_3'e_1$.

Finally let us consider the case when $q=7$. If $\xi(G)<6$, then by Theorems 2.6 and 2.4, $\kappa(J(G))\geqslant 3=\alpha(J(G))$. So, by Theorem 2.2, $J(G)$ is hamiltonian. If $\xi(G)=6$, then let $uv\in E(G)$ and $d(u)=d(v)=3$. Let $N(u)\backslash\{v\}=\{u_1,u_2\}$ and $N(v)\backslash\{u\}=\{v_1,v_2\}$. We distinguish two cases.

(i) $u_1=v_1$: Since G does not contain subgraph isomorphic to K_4^-, $u_2\neq v_2$. Let e_1 and e_2 be the remaining two edges which are incident with neither u nor v. Since G does not contain subgraph isomorphic to $K_3\circ K_1$, both e_1 and e_2 are not incident with u_1 ($=v_1$). Then we can obtain a Hamilton cycle of $J(G)$: $(uv)e_1(uu_1)(vv_2)(uu_2)(vu_1)e_2(uv)$.

(ii) $u_i\neq v_j$ for $1\leqslant i,j\leqslant 2$: Let e_1 and e_2 be the two edges as in (i). Without loss of generality, e_1 is not incident with v_2 and e_2 is not incident with u_1. Then we obtain a Hamilton cycle of $J(G)$: $(uv)e_1(vv_2)(uu_2)(vv_1)(uu_1)e_2(uv)$. □

4. Reducibility and semi-wheel

In this section, we introduce a few definitions, which are used in the proof of the main theorem. Let G be a graph of size q and $e=uv\in E(G)$. We call e is reducible, if $d(u)+d(v)\leqslant\lceil\frac{q+3}{2}\rceil$, equivalently, $d_J(e)\geqslant\lfloor\frac{q-1}{2}\rfloor$. Otherwise, e is said to be irreducible. We say G is reducible if there exists a reducible edge in G, otherwise, G is irreducible.

Lemma 4.1. *Let G be an irreducible graph of size q. We have*

(1) $\delta(G)\geqslant\lceil\frac{q+3}{2}\rceil-\Delta(G)+1$;
(2) *if* $q\geqslant 2\Delta(G)+1$ *and* $\Delta(G)\geqslant 4$, *then* $2q\geqslant\Delta(G)+2\lceil\frac{q+3}{2}\rceil+(\Delta-4)\delta(G)+2$.

Proof. (1) Let u be vertex of the minimum degree of G and $uw\in E(G)$. Since G is irreducible, we conclude that $d(u)+d(w)\geqslant\lceil\frac{q+3}{2}\rceil+1$. Thus the result follows.

Now we consider (2). Let $v\in V(G)$ with $d(v)=\Delta(G)$. Since $q\geqslant 2\Delta(G)+1$, we have

$$|E(G-v)|\geqslant\Delta(G)+1\geqslant\max\{5,\quad\Delta(G-v)+1\}.$$

This implies that there exist two edges in $G-v$, say wx and yz, which are not adjacent in G. Hence, $d(w)+d(x)\geqslant\lceil\frac{q+3}{2}\rceil+1$, $d(y)+d(z)\geqslant\lceil\frac{q+3}{2}\rceil+1$. Let p be the order of G, then

$$\begin{aligned}2q&\geqslant d(v)+d(w)+d(x)+d(y)+d(z)+(p-5)\delta(G)\\&\geqslant\Delta(G)+2\left(\left\lceil\frac{q+3}{2}\right\rceil+1\right)+(p-5)\delta(G)\\&\geqslant\Delta(G)+2\left\lceil\frac{q+3}{2}\right\rceil+(\Delta(G)-4)\delta(G)+2.\end{aligned}$$ □

Let $n\geqslant 4$ be an integer. Recall that wheel W_n of order n is the graph obtained from the cycle C_{n-1} of order $n-1$ by adding a new vertex and edges joining it to all the vertices of the cycle C_{n-1}. We define the concept of semi-wheel, denoted by SW_n, of order n. That is the graph obtained from the cycle C_{n-1} by adding a new vertex and edges joining it to $\lfloor\frac{n-1}{2}\rfloor$ pairwise nonadjacent vertices of C_{n-1}. The new edges are called the spokes of SW_n.

Let T be the set of vertices on C_{n-1} not incident with any spoke of SW_n. Then T is an independent set if n is odd, while there are exactly two vertices of T which are adjacent in C_{n-1} if n is even. Note that SW_n is not hamiltonian. However, if we add a chord to C_{n-1}, which joins any two vertices of T by a new edge e, then $SW_n + e$ is hamiltonian.

Lemma 4.2. *Let G be a graph of size $q \geqslant 2\Delta(G)$ and $\Delta(G) \geqslant 4$. If $e \in E(G)$ is reducible and $J(G-e)$ is hamiltonian, then $J(G)$ is hamiltonian.*

Proof. Let C be a Hamilton cycle of $J(G-e)$. If e is adjacent to two consecutive vertices of C in $J(G)$, then we are done. Note that $d_J(e) > \lfloor \frac{q-1}{2} \rfloor$ implies that e must have such two neighbors in $J(G)$, which consecutively appear on C, and $d_J(e) \geqslant \lfloor \frac{q-1}{2} \rfloor$ since e is reducible. So, now suppose that $d_J(e) = \lfloor \frac{q-1}{2} \rfloor$ and all neighbors of e are pairwise nonadjacent in C. Then e, $I_{J(G)}(e)$ and C formthe semi-wheel SW_q of order q. Let E_1 be the set of vertices on C not incident with any spoke of SW_q. To prove $J(G)$ is hamiltonian, it suffices to show that there exists an edge of $J(G)$, which is a chord of C joining two vertices of E_1. Suppose, on the contrary, there is no such an edge in $J(G)$. Let $E_2 = E_1 \cup \{e\}$, then $|E_2| = \lceil \frac{q+1}{2} \rceil$. If q is odd, then E_2 is an independent set of $J(G)$. Since $|E_2| = \frac{q+1}{2} \geqslant \Delta(G)+1 \geqslant 5 > 3$, the edge-induced subgraph $G[E_2]$ is a star. Then, $\Delta(G) \geqslant \Delta(G[E_2]) \geqslant \Delta(G)+1$, a contradiction. If q is even, then there are exactly two elements, say e_1 and e_2, in E_1, which are adjacent on C. Set $E_2' = E_2 \backslash \{e_1\}$ and $E_2'' = E_2 \backslash \{e_2\}$. Then both E_2' and E_2'' are independent sets of $J(G)$. Since $|E_2'| = |E_2''| = \frac{q}{2} \geqslant \Delta(G) \geqslant 4$, the edge-induced subgraphs $G[E_2']$ and $G[E_2'']$ are also stars. However, this is impossible since $|E_2' \cap E_2''| \geqslant 3$. $\square$

In the proof of the following lemma, we need a useful observation. Let $K_{m,m}^*$ be a bipartite graph with two partite set $\{e_1, \ldots, e_m\}$ and $\{e_1', \ldots, e_m'\}$, where $e_i e_j' \in E(K_{m,m}^*)$ if and only if $i \neq j$. Note that if $m \geqslant 3$, then for any edge $e_i e_j'$, there exists a Hamilton cycle of $K_{m,m}^*$ containing $e_i e_j'$.

Lemma 4.3. *Let G be a graph of size q. Then $J(G)$ is hamiltonian if one of the following conditions holds*:

(a) $q = 2\Delta(G) > \xi(G)$, $\Delta(G) \geqslant 4$, *and G contains two vertices u, v of degree $\Delta(G)$.*
(b) $q = 2\Delta(G)+1$, $\Delta(G) \geqslant 4$, *and* $\xi(G) = 2\Delta(G)$.
(c) $q = 8 > \xi(G)$, $\Delta(G) = 4$, *and G contains a subgraph isomorphic to K_4^- or $K_3 \circ K_1$, but contains no subgraph isomorphic to K_4.*
(d) $q = 9$, $\Delta(G) = 4$, *and G contains a subgraph isomorphic to K_4.*
(e) $q = 11$, $4 \leqslant \Delta(G) \leqslant 5$, *and G contains a subgraph isomorphic to K_5.*

Proof. (a) Since $\xi(G) < 2\Delta(G)$, u and v are not adjacent in G. Setting $I_G(u) = \{e_1, \ldots, e_m\}$ and $I_G(v) = \{e_1', \ldots, e_m'\}$, where $m = \Delta(G)$. We can assume that if e_i and e_j' are adjacent in G, then $i = j$. Therefore, if $i \neq j$, e_i and e_j' are adjacent in $J(G)$, and thus $J(G)$ contains a spanning subgraph isomorphic to $K_{m,m}^*$. So, $J(G)$ is hamiltonian.

(b) Since $\xi(G) = 2\Delta(G)$, let $e = uv \in E(G)$ with $d(u) = d(v) = \Delta(G)$. Similar to (a), let $I_G(u) \backslash \{e\} = \{e_1, \ldots, e_m\}$ and $I_G(v) \backslash \{e\} = \{e_1', \ldots, e_m'\}$, where $m = \Delta(G)-1$, and assume that if e_i and e'_j are adjacent in G, then $i = j$. Since $q = 2\Delta(G) + 1$, there are the only two edges, say e', e'', of G, which are neither incident with u nor v. Since $\Delta(G) \geqslant 4$, it is easy to find two edges e_{i_0} and e'_{j_0} with $i_0 \neq j_0$, which are not adjacent to e' and e'', respectively, in G. Thus $e_{i_0}e'$, $e''e'_{j_0}$, $e_{i_0}e'_{j_0} \in E(J(G))$. Set $T = (I_G(u)\backslash\{e\}) \cup (I_G(v)\backslash\{e\})$,then the induced subgraph $J(G)[T]$ has a spanning subgraph isomorphic to $K^*_{m,m}$. Let C be a Hamilton cycle of $J(G)[T]$ containing the edge $e_{i_0}e'_{j_0}$, then we can find a Hamilton cycle of $J(G)$, which is obtained from C by replacing the edge $e_{i_0}e'_{j_0}$ with the path $e_{i_0}e'ee''e'_{j_0}$.

(c) Suppose that G contains a subgraph $H \cong K_3 \circ K_1$. Let $V(H) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ and $E(H) = \{u_1u_2, u_2u_3, u_3u_1\} \cup \{u_1v_1, u_2v_2, u_3v_3\}$. Let $E(G)\backslash E(H) = \{e_1, e_2\}$. Since $q = 8$ and $\Delta(G) = 4$, then one vertex in $\{u_1, u_2, u_3\}$, say u_1, has degree 4 in G. Suppose, without loss of generality, that e_1 is incident with u_1. Then one of v_2 and v_3, say v_2, is not incident with e_1. Moreover, by $\xi(G) < 8$, we have u_1 is the only vertex of degree 4 in G. So, e_2 is not incident with any u_i's. Hence, $J(G)$ has a Hamilton cycle: $(u_1u_2)e_2(u_1u_3)(u_2v_2)e_1(u_2u_3)(u_1v_1)(u_3v_3)(u_1u_2)$.

Now suppose that G contains no subgraph isomorphic to $K_3 \circ K_1$, but has a subgraph $H \cong K_4^-$. Let $V(H) = \{u_1, u_2, u_3, u_4\}$, and suppose $u_1u_3 \notin E(G)$ since G contains no subgraph isomorphic to K_4. By $\Delta(G) = 4$ and $\xi(G) < 8 = q$, it is easy to see that G has only one vertex of degree 4, which must be in $V(H)$. By symmetry, we only need to consider two possible cases when $d(u_1) = 4$ or $d(u_2) = 4$. First assume that $d(u_1) = 4$, and let $N(u_1)\backslash\{u_2, u_4\} = \{v_1, v_2\}$. If $\{e\} = E(G)\backslash(E(H) \cup \{u_1v_1, u_1v_2\})$, then e is not incident with u_2 and u_4, since $\xi(G) < 8$. Then the following is a Hamilton cycle of $J(G)$: $(u_1u_2)e(u_1u_4)(u_2u_3)(u_1v_1)(u_2u_4)(u_1v_2)(u_3u_4)(u_1u_2)$.

Second, we assume $d(u_2) = 4$. Let v be in $N(u_2)$ other than u_1, u_3 and u_4 and $E(G)\backslash(E(H) \cup \{u_2v\}) = \{e_1, e_2\}$. Note that each of e_1 and e_2 is not incident with u_4. Since G contains no subgraph isomorphic to $K_3 \circ K_1$, if e_i, $i \in \{1, 2\}$, is incident with u_j for some $j \in \{1, 3\}$, then it must be incident with v_1. Hence, it suffices to consider the following three cases.

Case 1: $\{vu_1, vu_3\} \subseteq E(G)$. Then $G \cong W_5$, and we find a Hamilton cycle of $J(G)$: $(u_2u_1)(u_3v)(u_2u_4)(vu_1)(u_2u_3)(u_1u_4)(u_2v)(u_3u_4)(u_2u_1)$.

Case 2: Exactly one of $\{vu_1, vu_3\}$ is an edge of $E(G)$. Let $vu_1 = e_1 \in E(G)$. Then e_2 is not incident with any u_i's. We can find a Hamilton cycle of $J(G)$:

$(u_1u_2)e_2(u_2u_4)e_1(u_2u_3)(u_1u_4)(u_2v)(u_3u_4)(u_1u_2)$.

Case 3: Neither vu_1 nor vu_3 is in $E(G)$. We can find a Hamilton cycle of $J(G)$: $(u_1u_2)e_1(u_2u_4)e_2(u_2u_3)(u_1u_4)(u_2v)(u_3u_4)(u_1u_2)$.

(d) Suppose that H is a subgraph of G isomorphic to K_4. Let $V(H) = \{u_1, u_2, u_3, u_4\}$, where $d(u_1)=4$, and let v be the neighbor of u_1 other than u_i $(i=2, 3, 4)$. Let $E(G)\backslash(E(H) \cup \{u_1v\}) = \{e_1, e_2\}$. By virtue of (b), we assume that u_1 is the only vertex of degree 4 in G. Thus, e_i $(1 \leqslant i \leqslant 2)$ is not incident with any u_j's. Then the following is a Hamilton cycle of $J(G)$: $(u_1v)(u_2u_3)(u_1u_4)e_1(u_3u_4)(u_1u_2)e_2(u_1u_3)(u_2u_4)(u_1v)$.

(e) By the assumption, $G \cong K_5 + K_2$ if $\Delta(G) = 4$, and $G \cong K_5^+$ if $\Delta(G) = 5$. Since $K_5 + K_2$ can be obtained from K_5^+ by splitting the vertex of degree 5 in K_5^+, it suffices to prove that $J(K_5^+)$ is hamiltonian. To show this, let $V(G) = \{u_i \mid 1 \leqslant i \leqslant 6\}$, where $G[\{u_i \mid 1 \leqslant i \leqslant 5\}] \cong K_5$ and $u_1u_6 \in E(G)$. Now we obtain a Hamilton cycle of $J(G)$: $(u_1u_2)(u_3u_5)(u_2u_4)(u_1u_5)(u_2u_3)(u_1u_4)(u_2u_5)(u_1u_3)(u_4u_5)(u_1u_6)(u_3u_4)(u_1u_2)$. $\square$

5. The Proof of Theorem 1.1

First, we prove the sufficiency. Let G be a graph of size q. If G satisfies condition (1) or (2), or (3), then by Lemmas 2.8 and 3.2, $J(G)$ is not hamiltonian. Now suppose G satisfies condition (4), and that $3 \leqslant \Delta(G) \leqslant 4$. Otherwise, if $\Delta(G) > 4$, then $q \leqslant 8 < 2\Delta(G)$, we are done by Lemma 2.8. If $\Delta(G) = 3$, then, by Lemma 3.3, $J(G)$ is not hamiltonian. Now let $\Delta(G) = 4$, and assume that $q = 8$, and G contains a subgraph isomorphic to K_4. Let $E' = E(G) \backslash E(K_4)$, then we have $\omega(J(G) - E') = \omega(J(K_4)) = 3 > 2 = |E'|$. Also, $J(G)$ is not hamiltonian. Finally, if $G \cong K_5$, then $J(K_5)$ is exactly the petersen graph that is not hamiltonian.

Next, we prove the necessity of the theorem by contradiction. Suppose that there exists a graph G that does not satisfy any condition of Theorem 1.1 and whose jump graph $J(G)$ is not hamiltonian. Moreover, suppose that G is such a graph of minimum size q.

By the results in Section 3, we have $\Delta(G) \geqslant 4$. Since G does not satisfy condition (1), we have $q \geqslant 2\Delta(G)$, and $q \leqslant 3\Delta(G) - 2$ by Corollary 2.7.

Claim 1. *If $q = 2\Delta(G)$, then G has only one vertex, say u, with $d(u) = \Delta(G)$. Moreover, for any $e \in I_G(u)$, $J(G - e)$ is hamiltonian.*

In fact, if there exist two vertices $v, w \in V(G)$ such that $d(v) = d(w) = \Delta(G)$, then v and w are not adjacent in G since G does not satisfy condition (2). By (a) of Lemma 4.3, $J(G)$ is hamiltonian, a contradiction. So, let u be the only vertex of G with $d(u) = \Delta(G)$.

Now, we show that for any edge $e \in I_G(u)$, $J(G - e)$ is hamiltonian. If not, then let $e_0 \in I_G(u)$ such that $J(G - e_0)$ is not hamiltonian. By the minimality of G, $G - e_0$ must satisfy one of the conditions of Theorem 1.1. Let $G' = G - e_0$, then $\Delta(G') = \Delta(G) - 1 \geqslant 3$ and G' has size $q' = 2\Delta(G') + 1$. Hence, we have $q' = 7$, $\Delta(G') = 3$ and G' contains a subgraph isomorphic to K_4^- or $K_3 \circ K_1$. Therefore, $q = 8$ and $\Delta(G) = 4$. However, G has no subgraph isomorphic to K_4, otherwise, G satisfies condition (4). Thus $J(G)$ is hamiltonian by (c) of Lemma 4.3. A contradiction.

Claim 2. *If $q \geqslant 2\Delta(G) + 1$, then for any e, $J(G - e)$ is hamiltonian.*

Suppose, on the contrary, $e_0 \in E(G)$ such that $J(G - e_0)$ is not hamiltonian. Then, by the minimality of G and $q \geqslant 2 \times 4 + 1 = 9$, $G - e_0$ satisfies (5) or one of (4). Let $G' = G - e_0$. If G' satisfies (5), that is, $G' \cong K_5$, then, $q = 11$, G contains K_5 and $\Delta(G) = 4$ or 5. By (e) of Lemma 4.3, $J(G)$ is hamiltonian, a contradiction. If G' satisfies one condition in (4), then by $q \geqslant 9$, it follows that $|E(G')| = 8$, G' contains K_4 and $\Delta(G') = 3$ or 4. Therefore, $q = 9$, and $\Delta(G) = 4$ since $J(G)$ is not hamiltonian. However, by (d) of Lemma 4.3, $J(G)$ is hamiltonian. A contradiction.

Write $q = 2\Delta(G) + x$, then $0 \leqslant x \leqslant \Delta(G) - 2$. We consider two cases.

Case 1: $x = 0$.

By Claim 1, let u be the only vertex of degree $\Delta(G)$ in G, then for an edge $e \in I_G(u)$, $J(G - e)$ is hamiltonian. Since $J(G)$ is not hamiltonian, e is irreducible by Lemma 4.2. Hence, we have $d(v) \geqslant \lceil \frac{q+3}{2} \rceil - \Delta(G) + 1 = 3$ for each $v \in N(u)$, where $q = 2\Delta(G)$.

Then,

$$4\Delta(G) = 2q \geqslant d(u) + \sum_{v \in N(u)} d(v) \geqslant 4\Delta(G),$$

which implies that $V(G) = \{u\} \cup N(u)$ and $d(v) = 3$ for all $v \in N(u)$. If $\Delta(G) = 4$, then G is the wheel of order 5. By (c) of Lemma 4.3, $J(G) = J(W_5)$ is hamiltonian, a contradiction. If $\Delta(G) \geqslant 5$, we claim that the closure $cl(J(G))$ of $J(G)$ is a complete graph. In fact, since $d(v) = 3$ for each $v \in V(G) \backslash \{u\}$, $G - u$ is a union of disjoint cycles. Thus, for each $e \in E(G) \backslash I_G(u)$, there are respectively, exactly two edges of $I_G(u)$ and two edges of $E(G) \backslash I_G(u)$ adjacent to e in G. Hence, $d_J(e) = \Delta(G) - 2 + \Delta(G) - 3 = 2\Delta(G) - 5$. Since $2(2\Delta(G) - 5) = 4\Delta(G) - 10 \geqslant 2\Delta(G) = q$, the subgraph induced by $E(G) \backslash I_G(u)$ in $cl(J(G))$ is a clique, and further, it is straightforward to verify that $cl(J(G)))$ is a complete graph. So, $J(G)$ is hamiltonian, a contradiction.

Case 2: $x \geqslant 1$.

By Claim 2 and Lemma 4.2, G is irreducible. Hence, it follows by (1) and (2) of Lemma 4.1,

$$2q \geqslant \Delta(G) + 2\left\lceil \frac{q+3}{2} \right\rceil + (\Delta(G) - 4)\left(\left\lceil \frac{q+3}{2} \right\rceil - \Delta(G) + 1\right) + 2.$$

Taking $q = 2\Delta(G) + x$ into the inequality above, we obtain

$$2(x+1) \geqslant (\Delta(G) - 2)\left\lceil \frac{x+3}{2} \right\rceil. \qquad (*)$$

Hence, $\Delta(G) \leqslant 5$. By $q \leqslant 3\Delta(G) - 2$, we may assume that $x \leqslant 2$ if $\Delta(G) = 4$, and $x \leqslant 3$ if $\Delta(G) = 5$. However, for each value of $x = 1, 2, 3$, it follows that $\Delta(G) \leqslant 4$ from the inequality $(*)$. Hence, it remains only two special cases to consider.

Subcase 2.1: $\Delta(G) = 4$ and $q = 2\Delta(G) + 1 = 9$.

Since G is irreducible, from Lemma 4.1 we have $\delta(G) \geqslant 3$. By the similar argument as in the proof of Case 1, we have $\delta(G) = 3$, and the order of G is 5. So, $G \cong K_5^-$, and $J(K_5^-)$ is isomorphic to the graph obtained from the petersen graph by deleting a vertex. Thus $J(G)$ is hamiltonian since the petersen graph is hypohamiltonian. (A graph G is said to be hypohamiltonian if G is not hamiltonian and for any $v \in V(G)$, $G - v$ is hamiltonian.)

Subcase 2.2: $\Delta(G) = 4$ and $q = 2\Delta(G) + 2 = 10$.

Since G is irreducible, by Lemma 4.1 we have $\delta(G) \geqslant 4$, and thus G is 4-regular. Since $q = 10$, it follows that the order of G is 5, and so $G \cong K_5$. This contradicts our choice of G that does not satisfy any condition of our theorem.

This completes the proof of Theorem 1.1. $\square$

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